exos : Thlafay . free . fr
mot de passe Cauchy

chapitre 7

Nombres complexes

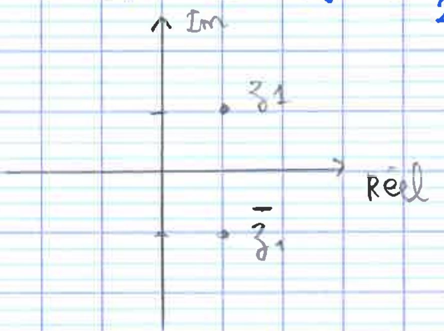
Exemple p 82 :

$1z^2 - 3z + 4 = 0 \quad (1)$

$\Delta = 9 - 16 = -7 \quad = i^2 7$

$z_1 = \dfrac{3 - i\sqrt{7}}{2} \qquad z_2 = \dfrac{3 + i\sqrt{7}}{2}$

$\Rightarrow (1) = 1 \times \left(z - \dfrac{3 - i\sqrt{7}}{2}\right)\left(z - \dfrac{3 + i\sqrt{7}}{2}\right)$



$\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\| = \sqrt{x^2 + y^2}$

$|x + iy| = \sqrt{x^2 + y^2}$

module $\rightarrow |x| = \left(\sqrt{x^2 + y^2}\right)$

$\sqrt{x^2 + 0^2} = \sqrt{x^2} = |x|$ valeur absolue

p 82 $\quad z\bar{z} = |z|^2$

Preuve : $z = x + iy$

$\bar{z} = x - iy$

$z\bar{z} = (x + iy)(x - iy)$

$= x^2 - (iy)^2$

$= x^2 - i^2 y^2 = x^2 + y^2 = |z|^2$

Exemple :

$$z = \frac{1+3i}{1-i} + i \Leftrightarrow z = \frac{(1+3i)(1+i)}{1-i^2} + i$$

$$= \frac{1+i+3i+3i^2}{2} = \frac{1+4i+3i^2}{2} + i = \frac{-2+4i}{2} + i$$

$$= -1+3i$$

$\mathrm{Re}(z) = -1 \qquad \mathrm{Im}(z) = 3$

Supposons $z = x+iy \quad \bar{z} = x-iy$ où $(x,y) \in \mathbb{R}^2$

$$x+iy = 1+i-2i(x-iy) + \frac{i(1+i)}{2}$$

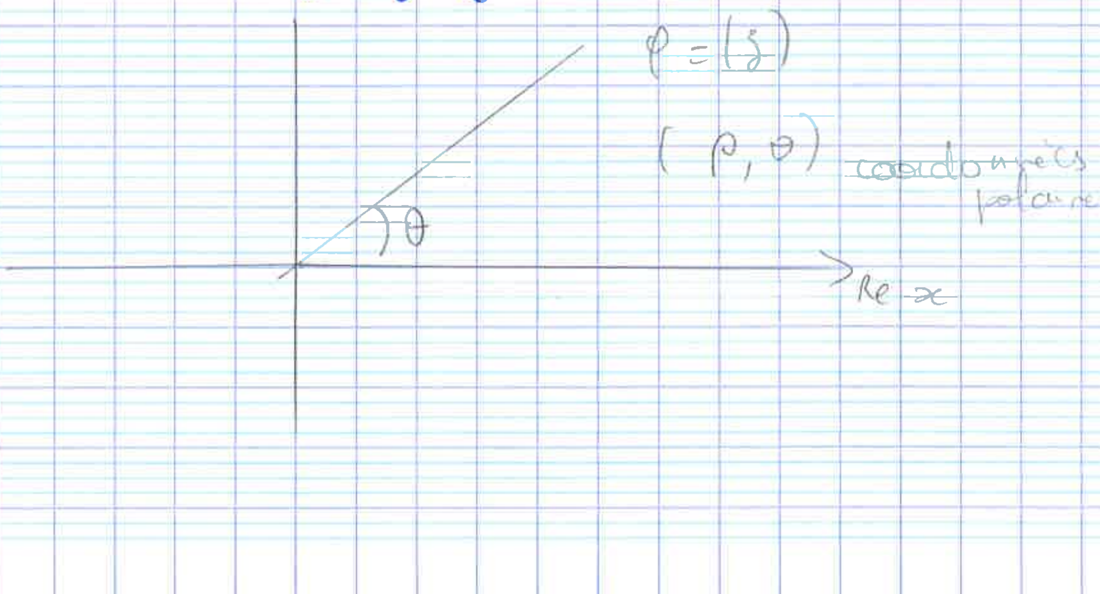
$$x+iy = \frac{1}{2} + \frac{3i}{2} - 2ix - 2y.$$

On identifie partie réelle puis partie imaginaire

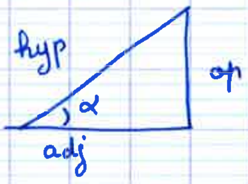
$$\begin{cases} x = \frac{1}{2} - 2y \\ y = \frac{3}{2} - 2x \end{cases} \qquad \begin{cases} x = \frac{1}{2} - 2y. \\ y = \frac{3}{2} - 1 + 4y \end{cases}$$

$$\begin{cases} x = \frac{1}{2} - 2y \\ -\frac{1}{2} = 3y \end{cases} \qquad \begin{cases} x = \frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6} \\ y = -\frac{1}{6} \end{cases}$$

Conclusion : $z = \frac{5}{6} - \frac{i}{6}$



$\alpha \in [0, 2\pi]$



$\cos \alpha = \dfrac{\text{côté } \cancel{\text{hyp}} \text{ adj}}{\text{hyp}}$

$\sin \alpha = \dfrac{\text{coté opp}}{\text{hyp}}$

Propriété $\cos \alpha \in [-1 ; 1]$

$\sin \alpha \in [-1 ; 1]$

$\tan \alpha = \dfrac{\text{coté opp}}{\text{coté adj}} = \dfrac{\sin \alpha}{\cos \alpha}$

Th pythagore

$(\text{adj})^2 + (\text{opp})^2 = (\text{hyp})^2$ $\quad \cos^2 \alpha + \sin^2 \alpha = 1$

Sur le $\mathcal{C}$ de rayon 1 centré en O

$\cos \alpha = \mathrm{Re}(z)$

$\sin \alpha = \mathrm{Im}(z)$

$\cos \alpha = 1 \quad \sin 0 = 0$

$\cos \frac{\pi}{2} = 0 \quad \sin \frac{\pi}{2} = 0$

Pour $\frac{\pi}{4}$ on est sur la 1ère bissectrice donc $\cos \frac{\pi}{4} = \sin \frac{\pi}{4}$

$\cos^2 \frac{\pi}{4} + \sin^2 \frac{\pi}{4} = 1$

$2 \cos^2 \frac{\pi}{4} = 1 \Leftrightarrow \cos^2 \frac{\pi}{4} = \frac{1}{2} \Leftrightarrow \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ car ça doit ê positif

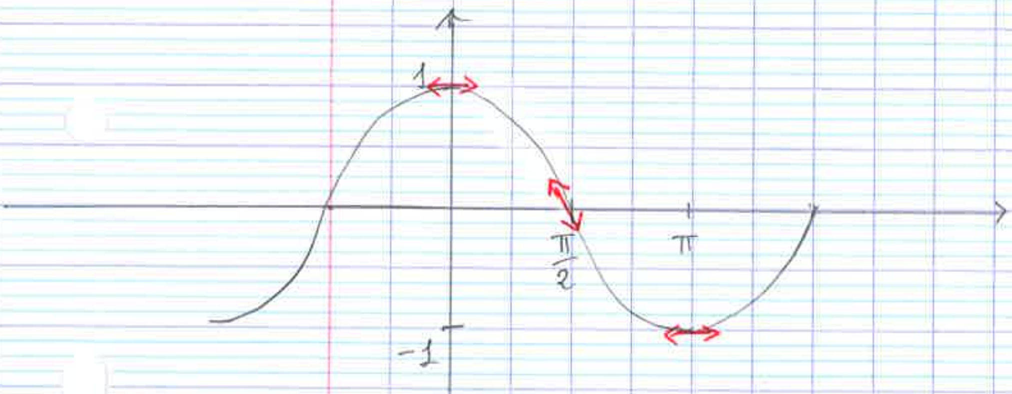
α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin \alpha$	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{4}}$
$\cos \alpha$	$\sqrt{\frac{4}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{0}{4}}$
$\sin \alpha$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos \alpha$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0

cos α est paire cos α = cos (– x)
sin x est impaire sin (–x) = – sin x
tan (– x) = – tan x tan est impaire
L'étude suffit sur [0; π] pour les fct° trigo

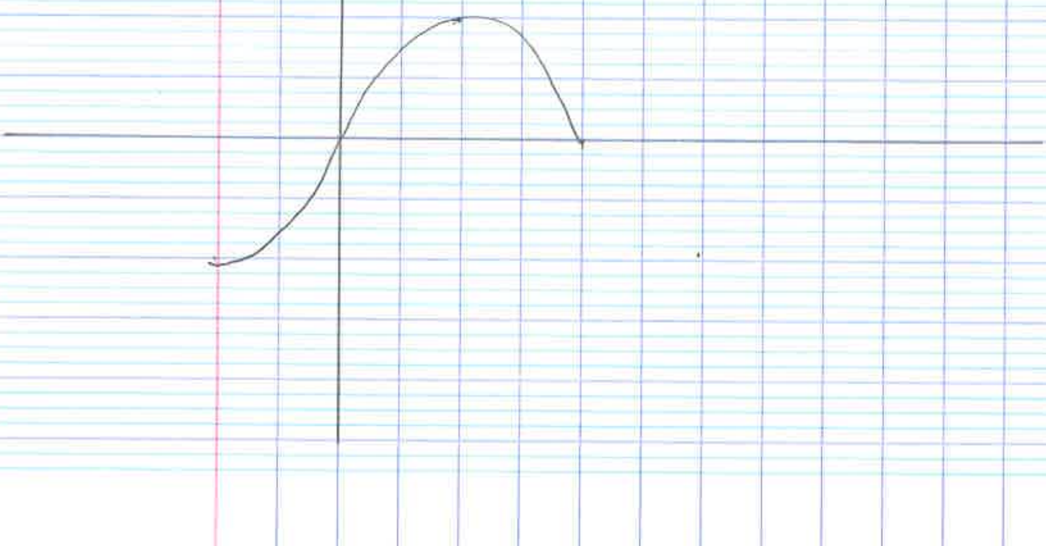
Etude de cos x

$D_f = \mathbb{R}$
$D_E = [0, \pi]$
$(\cos x)' = -\sin x$

	0			π
f'(x) = – sin x	0		–	0
f(x) = cos x	1	↘	0 ↘	–1



$\sin x = \cos\left(x - \frac{\pi}{2}\right)$



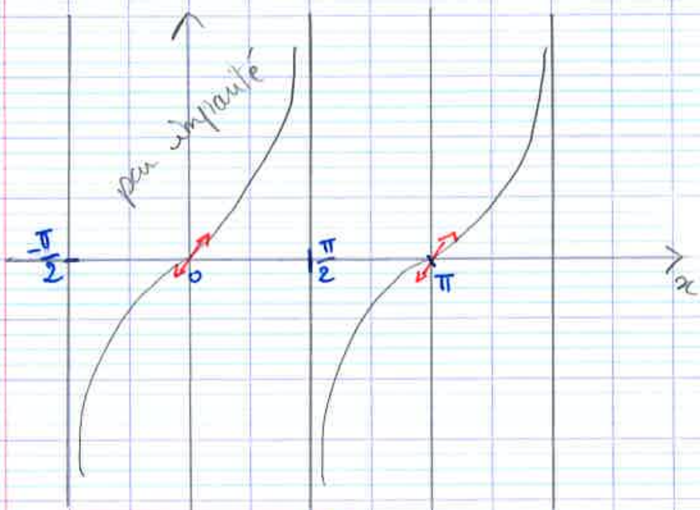
$$\tan x = \frac{\sin x}{\cos x}$$

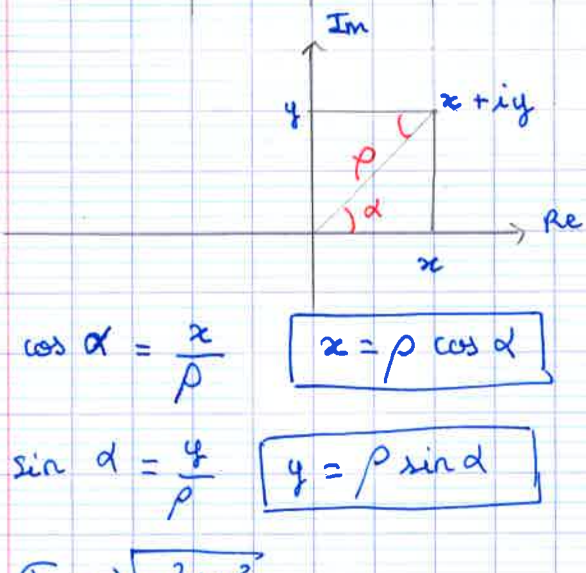
$$(\tan x)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$\tan 0 = 0$ $\quad \lim\limits_{x \to \frac{\pi}{2}^-} \tan x = \frac{1}{0^+} = +\infty$

$(\tan x)'(0) = 1$ $\quad \lim\limits_{x \to \frac{\pi}{2}^+} \tan x = \frac{1}{0^-} = -\infty$

$\tan \pi = 0$





$\cos \alpha = \frac{x}{\rho}$ $\quad \boxed{x = \rho \cos \alpha}$

$\sin \alpha = \frac{y}{\rho}$ $\quad \boxed{y = \rho \sin \alpha}$

$\sigma = \sqrt{x^2 + y^2}$

$\rho (\cos \alpha + i \sin \alpha) = x + iy.$

$\cos \alpha = \frac{x}{\rho}$

$\sin \alpha = \frac{y}{\rho}$ $\qquad \rho$ est appelé argument du complexe

$z = \sqrt{2} + i\sqrt{2}$

$|z| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4} = 2$

$z = 2\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right)$

$z = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \Rightarrow \mathrm{Arg}\,(z) = \frac{\pi}{4}\,(2\pi)$

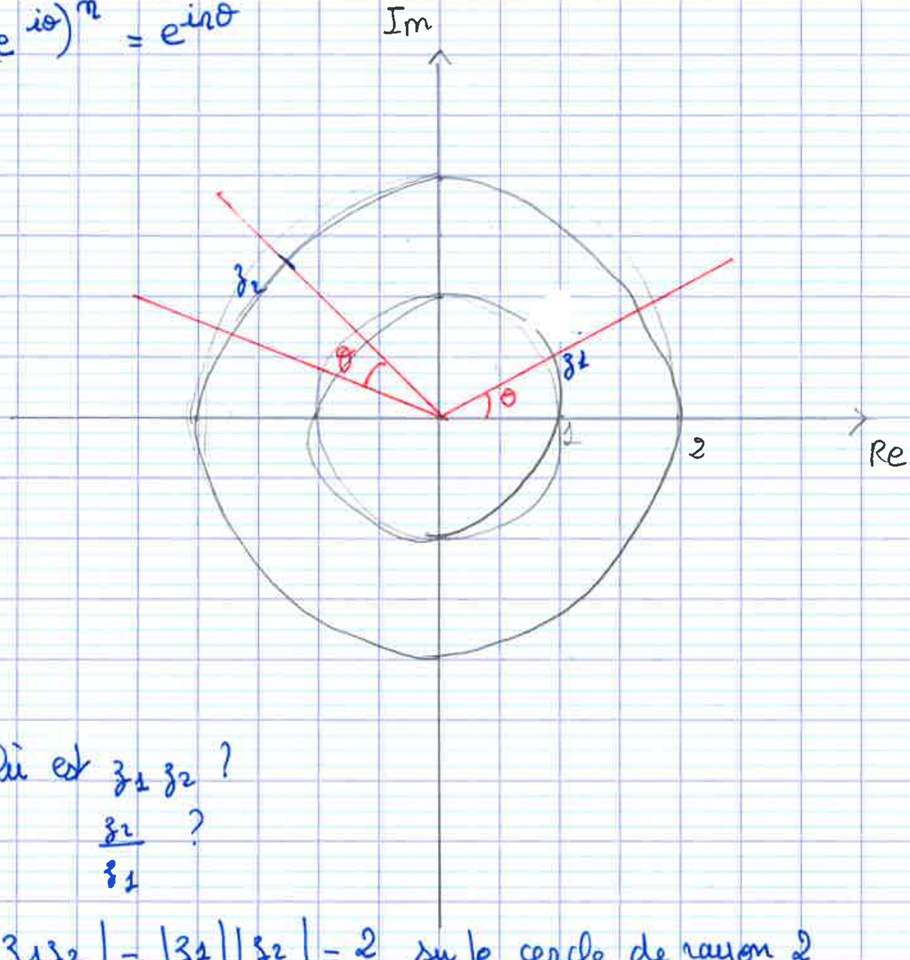
coordonneés polaires $\rho = 2 \quad \theta = \frac{\pi}{4}\,(2\pi)$

$z = \rho e^{i\theta} = \rho\,(\cos\theta + i\sin\theta)$

On pose : $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

$e^{i(\theta+\varphi)} = e^{i\theta} e^{i\varphi}$

$(e^{i\theta})^n = e^{in\theta}$



Où est $z_1 z_2$?

$\frac{z_2}{z_1}$?

$|z_1 z_2| = |z_1||z_2| = 2$ sur le cercle de rayon 2

$\mathrm{Arg}\,(z_1 z_2) = \mathrm{Arg}\,(z_1) + \mathrm{Arg}\,(z_2)$

$$\frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{\cos\theta + i\sin\theta + (\cos(-\theta) + i\sin(\theta))}{2i}$$

$$= \frac{\cos\theta - \cos\theta + i\sin\theta + i\sin\theta}{2i}$$

$$= \frac{2i\sin\theta}{2i} = \sin\theta$$

Formule d'Euler :

$$* \ \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)}{2}$$

$$= \frac{2\cos\theta + i\sin\theta - i\sin\theta}{2}$$

$$= \cos\theta$$

1^{ere} méthode : $z = x + iy$

$\bar{z} = x - iy$.

$(x+iy)^2 = x - iy \iff x^2 - y^2 + 2ixy = x - iy$

$$\begin{cases} x^2 - y^2 = x \\ 2xy = -y \end{cases} \qquad \begin{cases} x^2 - y^2 = x \quad (1) \\ y(2x+1) = 0 \quad (2) \end{cases}$$

1^{er} cas : $y = 0$

$(1) \iff x^2 = x$ donc $x = 0$ ou $x = 1$

$0 = 0 + i0$ ou $1 = 1 + i0$

2^e cas : $x = -\frac{1}{2}$

$(1) \iff \frac{1}{4} - y^2 = -\frac{1}{2} \iff y^2 = \frac{3}{4} \quad y = \frac{\sqrt{3}}{2}$ ou $-\frac{\sqrt{3}}{2}$

$e^{i\frac{2\pi}{3}} = -\frac{1}{2} + \frac{i\sqrt{3}}{2} \qquad -\frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{-2i\pi/3}$

On a 4 solutions $0, 1; e^{i\frac{2\pi}{3}}; e^{-2i\frac{\pi}{3}} = e^{4i\frac{\pi}{3}}$

2^e méthode $z^2 = \bar{z}$

$z = \rho e^{i\theta}$ $\bar{z} = \rho e^{-i\theta}$

$z^2 = (\rho e^{i\theta})^2 = \rho e^{-i\theta}$

$\rho^2 e^{2i\theta} = \rho e^{-i\theta}$ en égalisant module et arg.

$\begin{cases} \rho^2 = \rho \\ 2\theta \equiv -\theta \ [2\pi] \end{cases}$ $\begin{cases} \rho^2 = \rho \\ 3\theta \equiv 0 \ [2\pi] \end{cases}$

1^er cas $\rho = 0 \Rightarrow 1$ solution : 0

2^e cas $\rho = 1$

$3\theta = 0 \ [2\pi] \Leftrightarrow 3\theta = 0 + 2k\pi$ avec $k \in \mathbb{Z}$

$\theta \equiv 0 \left[\frac{2\pi}{3}\right] \Leftrightarrow \theta = 0 + \frac{2\pi k}{3}$ avec $k \in \mathbb{Z}$

$e^{i\theta} = 1$; $e^{2i\pi/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$e^{4i\frac{\pi}{3}} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

les solutions sont : $0, 1, e^{2i\pi/3}, e^{4i\pi/3}$

1^ère méthode : $z = x + iy$.

$(x+iy)^3 = x - iy$

$x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3 = x - iy$.

$x^3 - 3xy^2 + 3x^2 y i - iy^3 = x - iy$.

$\begin{cases} x^3 - 3xy^2 = x \\ 3x^2 y - y^3 = -y \end{cases}$ ⚠ compliqué

2^e méthode : $z = \rho e^{i\theta}$

$(\rho e^{i\theta})^3 = \rho e^{-i\theta}$

$\rho^3 e^{3i\theta} = \rho e^{-i\theta}$

$\rho^3 = \rho$

$3\theta = -\theta \ [2\pi]$

1^er cas $\rho = 0$

2^e $\rho = 1$

$4\theta \equiv 0 \ (2\pi)$

$\theta \equiv 0 \ (\pi/2)$

Les solut° sont $0, 1e^{i0}, 1e^{i\pi/2}, 1e^{i\pi}, 1e^{i3\pi/2}$

$S = \{0, 1; i, -1, -i\}$

$z^n = re^{i\alpha}$

$\Leftrightarrow \rho^n e^{ni\theta} = re^{i\alpha}$

Gabri

P84 Exemple : Cherchez les racines 3^e de 8 :

les Racines 2° de 4 → ~~sont~~ 2 et −2

Preuve $2^2 = 4$

$(-2)^2 = 4$

$\sqrt[3]{8} = 2 = 8^{1/3}$

Soit $z = \rho e^{i\theta}$ une racine 3^e de 8

$\rho^3 e^{3i\theta} = 8 = 8e^{i0}$

$\begin{cases} \rho^3 = 8 \\ 3\theta = 0 \ [2\pi] \end{cases}$ $\begin{cases} \rho = 2 \\ \theta = 0 [2\pi/3] \end{cases}$

$2e^{i0}$; $2e^{2i\pi/3}$; $2e^{4i\pi/3}$

$z_1 = 2$

$z_2 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= -1 + i\sqrt{3}$

$z_3 = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$= 2\left\{\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right\}$

$= 2\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$

$= -1 - i\sqrt{3}$

⚠ $\sqrt[3]{-8} = -2$ se définit car ∃ une racine réelle mais $\sqrt{-2}$ ne se définit pas.

mais $(-8)^{1/3} \neq -2$

pas égal

$P(z) = a_n z^n + \dots + a_0 \quad a_i \in \mathbb{Z}$

$P(z) = \alpha_n X^n + \dots + \alpha_0 \quad \alpha_i \in \mathbb{R}$

tous les polynomes réels ne sont pas factorisables dans $\mathbb{R}$

Exemple $P(X) = X^2 + X + 1$

$\Delta = 1 - 4 = -3 < 0$

$P(X) \neq (X - \alpha_1)(X - \alpha_2)$

Par contre dans $\mathbb{C}$: tous les polynomes à coefficients complexes sont factorisables

$P(z) = a_n \prod_{i=n}^{n} (z - \lambda_i)$ avec λ_i complexe.

Supposons que l'on cherche les racines du complexe $\alpha + i\beta$

Méthode 1 : $z = x + iy$ tq $z^2 = \alpha + i\beta$

$(x + iy)^2 = \alpha + i\beta$

$x^2 - y^2 + 2ixy = \alpha + i\beta$

$\begin{cases} 2xy = \beta \\ x^2 - y^2 = \alpha \end{cases}$

Méthode 2 $\Delta = \alpha + i\beta = \rho e^{i\theta}$

z l'une des racines est alors $z = \sqrt{\rho} e^{i\theta/2}$ $\boxed{(re^{i\varphi})^2 = \rho e^{i\theta}}$

$P(z) = z^2 - (1+3i)z + 2i - 2 = 0$

$\Delta = (1+3i)^2 - 4(2i - 2)$

$= 1^2 + (3i)^2 + 6i - 8i + 8$

$= 1 - 9 - 2i + 8$

$= -2i$

$= 2(-i)$

$= 2e^{-i\pi/2}$

$\begin{cases} r^2 = \rho \\ 2\varphi = \theta \ (2\pi) \end{cases}$

$\begin{cases} r = \sqrt{\rho} \\ \varphi = \frac{\theta}{2} \ [\pi] \end{cases}$

$\arg(z^{1/2}) = \frac{1}{2} \arg z$

Notons δ l'une des racines de Δ

$\delta = \sqrt{2} e^{-i\pi/4}$

$\delta = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right)\right)$

$= \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = 1 - i \ (\text{"}\sqrt{\Delta}\text{"})$

$$z_1 = \frac{1+3i-(1-i)}{2} = 2i$$

$$z_2 = \frac{1+3i+1-i}{2} = 1+i$$

$$P(z) = 1z^2 - (1+3i)z + 2i - 2$$
$$= 1(z-2i)(z-1-i)$$

$i^2 = -1 \quad e^{i\theta} = \cos\theta + i\sin\theta$

$$\cos^2\theta + \sin^2\theta = 1$$

$$e^{-ia} = \cos(-a) + i\sin(-a)$$
$$= \frac{1}{e^{ia}} = \frac{1}{\cos a + i\sin a} \qquad = \cos a - i\sin a$$
$$= \frac{\cos a - i\sin a}{(\cos a + i\sin a)(\cos a - i\sin a)}$$

$\cos(-a) = \cos a$

$\sin(-a) = -\sin a$

$$e^{i(a+\pi/2)} = e^{ia}e^{i\pi/2}$$
$$= (\cos a + i\sin a)\, i$$
$$= -\sin a + i\cos a$$
$$= \cos\left(a+\frac{\pi}{2}\right) + i\sin\left(a+\frac{\pi}{2}\right)$$

$$\begin{cases} \cos\left(a+\frac{\pi}{2}\right) = -\sin a \\ \sin\left(a+\frac{\pi}{2}\right) = \cos a \end{cases}$$

$$e^{i(a+b)} = \cos(a+b) + i\sin(a+b)$$
$$= e^{ia}e^{ib} = (\cos a + i\sin a)(\cos b + i\sin b)$$
$$= \cos a\cos b - \sin a\sin b + i(\cos a\sin b + \sin a\cos b)$$

$\cos(a+b) = \cos a\cos b - \sin a\sin b$.

$\sin(a+b) = \cos a\sin b + \sin a\cos b$.

$\Rightarrow \sin(2a) = 2\cos a$

$P(z) = \frac{1}{4}z^2 + (1+\sqrt{3}i)z + 1 - \sqrt{3}i$

$\Delta = (1+\sqrt{3}i)^2 - \frac{4}{4}(1-\sqrt{3}i)$.

$= 1-3+2\sqrt{3}i - 1+\sqrt{3}i$

$= -3+3\sqrt{3}i$

$|\Delta| = \sqrt{3^2+(3\sqrt{3})^2} = \sqrt{9+27} = \sqrt{36} = 6$

$\Delta = -3+3\sqrt{3}i$

$= 6\left(-\frac{3}{6} + \frac{3\sqrt{3}}{6}i\right)$

$= 6\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 6e^{2i\pi/3}$

$\delta = \sqrt{6}e^{i\pi/3} = \sqrt{6}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= \sqrt{6}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right).$

$z_1 = \frac{-1-i\sqrt{3}-\sqrt{6}/2 - 3i\sqrt{2}/2}{\frac{1}{2}}$

$z_1 = -2-2i\sqrt{3}-\sqrt{6}+3i\sqrt{2}$

$\boxed{z_1 = -2-\sqrt{6}+i(3\sqrt{2}-2\sqrt{3})}$

$z_2 = \frac{-1-i\sqrt{3}+\sqrt{6}/2+3i\sqrt{2}/2}{1/2}$

$\boxed{z_2 = -2+\sqrt{6}+i(3\sqrt{2}-2\sqrt{3})}$

$z_1 = 0$

$z_2 = 1$

$z_3 = \frac{1}{2}\left(-1+2^{1/4}\cos\frac{3\pi}{8}\right) + \frac{i}{2}\left(1+2^{1/4}\sin\frac{3\pi}{8}\right)$

$z_4 = \frac{1}{2}\left(-1-2^{1/4}\cos\frac{3\pi}{8}\right) + \frac{i}{2}\left(1-2^{1/4}\sin\frac{3\pi}{8}\right)$

p86 $\quad e^{i\left(\frac{a+b}{2}\right)}e^{i\left(\frac{a-b}{2}\right)} = e^{ia} = \cos a + i\sin a$

$= \left(\cos\frac{a+b}{2} + i\sin\frac{a+b}{2}\right)\left(\cos\frac{a-b}{2} + i\sin\frac{a-b}{2}\right)$

En identifiant la partie réelle

$\cos a = \cos\frac{a+b}{2}\cos\frac{a-b}{2} - \sin\frac{a+b}{2}\sin\frac{a-b}{2} \quad \forall b$.

$\cos b = \cos\frac{b+a}{2}\cos\frac{b-a}{2} - \sin\frac{b+a}{2}\sin\frac{b-a}{2}$

En sommant $\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{b-a}{2}$